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ELLIPSOIDAL APPROXIMATION OF ATTAINABILITY SETS OF A LINEAR SYSTEM WITH INDETERMINATE MATRIX†

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Linear dynamical systems described by finite-difference or differential equations are considered. It is assumed that the matrix of the system is either completely known or is subject to uncontrollable perturbations, so that each element is known only to within a certain possible interval. Outer approximations, by means of ellipsoids, are constructed for the attainability sets of such systems. The equations of evolution of the approximating ellipsoids are obtained. An example is presented.

This paper continues previous studies [1–3], which were concerned with additive perturbations. The present investigation centres on the more complicated case of multiplicative perturbations (the perturbations are multiplied by phase coordinates).

1. FORMULATION OF THE PROBLEM FOR DISCRETE TIME SYSTEMS

We first consider a discrete time system (multistage process) described by the linear finite-difference equations

$$x(t_{i+1})=C(t_i)x(t_i)+f(t_i),\quad t_0<t_1,\ldots,\quad i=0,\ 1,\ldots \tag{1.1}$$

where x is the n-vector of phase coordinates of the system, C is an $n\times n$ matrix and f is an n-vector. The vectors and matrices $x(t_i)$, $C(t_i)$, $f(t_i)$ are defined at given discrete instants of time t_i ($i = 0, 1, \ldots$), the vector function $f(t_i)$ is assumed to be a given function of time, and the matrix $C(t_i)$ contains an indeterminate component and is expressed as

$$C(t_i)=C_0(t_i)+C_1(t_i),\quad i=0,\ 1,\ \ldots \tag{1.2}$$

where $C_0(t_i)$ is a given non-singular matrix depending on time, but $C_1(t_i)$ is unknown—either because it is subject to perturbations or because the structure of the system is incompletely known. It is assumed that the elements $c_{jk}(t_i)$ of $C_1(t_i)$ are bounded in absolute value

$$|c_{jk}(t_i)|\leqslant b_{jk}(t_i),\quad j,\ k=1,\ \ldots,\ n;\quad i=0,\ 1,\ \ldots \tag{1.3}$$

where $b_{jk}(t_i)$ are given numbers.

The initial state of system (1.1) may also not be accurately known; all that is known is a set M containing it

$$x(t_0)\in M,\quad M\subset R^n \tag{1.4}$$

The attainability set $D(t_i, t_0, M)$ of system (1.1), $i\geqslant 0$, is defined as the set of points $x(t_i)$ that are the ends of all phase trajectories $x(\cdot)$ of the system admitted by conditions (1.2)–(1.4).

This set possesses the following evolution (semigroup) property [2, 3]

$$D(t_i,\ t_0,\ M)=D(t_i,\ t_j,\ D(t_j,\ t_0,\ M)),\quad 0\leqslant j\leqslant i \tag{1.5}$$

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The attainability set characterizes the possible spread of trajectories of the system under the effect of the perturbations or the uncertainty factors. As is well known, the exact construction of attainability sets is generally fraught with difficulties. This explains why it is important to have fairly simple and effective outer approximations for the set.

As approximating sets, we shall use ellipsoids, which were used in the previously considered case of additively introduced perturbations. In this paper we will consider the more complicated case of multiplicative perturbations, which has been studied before [4] and occurs not infrequently in applications.

Such applications include mechanical and electrical systems with perturbed or incompletely known stiffness coefficients, mechanical or electrical resistance, inductance, capacitance, and so on, as well as linear controllable systems with inaccuracies in the realization of the amplification factors. Ellipsoidal approximations possess several advantages: simplicity, smoothness of the boundary, invariance under linear transformations, and so on [2, 3].

Following [1–3], we introduce the notation $E(a, Q)$ for an n-dimensional ellipsoid

$$E(a,\ Q)=\{x:(Q^{-1}(x-a),\ (x-a))\leqslant 1\} \tag{1.6}$$

where a is the n-vector of the centre of the ellipsoid, Q is a symmetric positive definite $n\times n$ matrix and $(\cdot,\cdot)$ is the scalar product of vectors. Note that as $Q\to 0$ the ellipsoid (1.6) shrinks to the point $x=a$.

The problem of the outer ellipsoid approximation of attainability sets may be formulated as follows.

Problem 1. It is required to find a vector-valued function $a(t_i)$ and a matrix-valued function $Q(t_i)$ such that

$$D(t_i,\ t_0,\ M)\subset E(a(t_i),\ Q(t_i)),\quad \forall i=0,\ 1,\ \ldots \tag{1.7}$$

2. FORMULATION OF THE PROBLEM FOR CONTINUOUS TIME SYSTEMS

We now consider a linear system of ordinary differential equations with initial condition

$$\dot{x}=C(t)x+f(t),\quad t\geqslant s;\quad x(s)\in M,\quad M\subset R^n \tag{2.1}$$

where x is the n-vector of phase coordinates, the dot denotes differentiation with respect to time t, C is an $n\times n$ matrix, f is an n-vector and M a given initial set. The functions $C(t)$ and $f(t)$ are such that the initial-value problem (2.1) has a solution for any initial vector $x(s)$; it is sufficient to assume that these functions are piecewise-continuous for $t\geqslant s$. In that case the function $f(t)$ is given for $t\geqslant s$ and the function $C(t)$ may be expressed as

$$C(t)=C_0(t)+C_1(t) \tag{2.2}$$

where the matrix $C_0(t)$ is given for $t\geqslant s$ and the matrix $C_1(t)$ is unknown. The elements $c_{jk}(t)$ of $C_1(t)$ satisfy inequalities similar to (1.3)

$$|c_{jk}(t)|\leqslant b_{jk}(t),\quad j,\ k=1,\ \ldots,\ n,\quad t\geqslant s \tag{2.3}$$

where $b_{jk}(t)$ are given non-negative functions of time, defined for $t\geqslant s$.

The attainability set $D(t, s, M)$ of system (2.1) for $t\geqslant s$ is defined as the set of all points $x(t)$ that are ends of the phase trajectories $x(\cdot)$ of the system at time t allowed by conditions (2.1)–(2.3). The attainability set possesses an evolution property [2, 3] similar to (1.5)

$$D(t,\ s,\ M)=D(t,\ \tau,\ D(\tau,\ s,\ M)),\quad \tau\in[s,\ t] \tag{2.4}$$

The problem of constructing an outer ellipsoidal approximation to the attainability sets of system (2.1)–(2.3) is analogous to Problem 1.

Problem 2. Find a vector function $a(t)$ and a matrix function $Q(t)$ such that, for all $t\geqslant s$

$$D(t, s, M) \subset E(a(t), Q(t)), \quad t \geqslant s \tag{2.5}$$

Problems 1 and 2 obviously have more than one solution: any ellipsoid containing an ellipsoid solution is also a solution. It is therefore natural to ask whether the solution can be minimized in the sense of some optimum criterion characterizing the "size" of the approximating ellipsoids, for example their volume, sum of squared semi-axes, or the like. This was the approach adopted previously [1–3] for additive perturbations. In this paper the problem of optimizing the approximating ellipsoids will not be solved, but in the construction we will use certain optimal operations on ellipsoids [1–3].

3. TRANSFORMATIONS OF ELLIPSOIDS

Returning to Problem 1, let us assume that the required ellipsoid $E(a(t_i), Q(t_i))$ satisfying the inclusion (1.7) at time $t = t_i$ has been constructed. By (1.1) and (1.2), we have

$$x(t_{i+1}) = x_1 + x_2; \quad x_1 = C_0(t_i)x(t_i) + f(t_i), \quad x_2 = C_1(t_i)x(t_i) \tag{3.1}$$

In order to construct an ellipsoid $E(a(t_{i+1}), Q(t_{i+1}))$ containing the vector $x(t_{i+1})$, it will suffice, according to (3.1), to perform the following three operations:

1. construct an ellipsoid containing the vector x_1;
2. construct an ellipsoid containing the vector x_2;
3. construct an ellipsoid containing the vector sum $x_1 + x_2$, where each vector in the sum is known to belong to a certain ellipsoid.

We will consider each of these operations separately.

The first operation reduces to an affine transformation of ellipsoids. Suppose that, for some n-vector x, it is known that $x \in E(a, Q)$, where a is the n-vector representing the centre of the ellipsoid and Q is a symmetric positive definite $n \times n$ matrix; then [1–3]

$$Ax + b \in E(Aa + b, \ AQA^T) \tag{3.2}$$

where A is an arbitrary non-singular $n \times n$ matrix, b is an n-vector and the superscript T denotes transposition. By (3.2), we obtain the following inclusion for the vector x_i of (3.1)

$$x_1 \in E(C_0(t_i)a(t_i) + f(t_i), \ C_0(t_i)Q(t_i)C_0^T(t_i)) \tag{3.3}$$

The second operation (construction of an ellipsoid containing x_2) reduces to solving the following auxiliary problem.

Problem 3. Find an ellipsoid containing the set

$$\Omega = \{y : y = Cx, \ C \in \Sigma, \ x \in E(a, Q)\} \tag{3.4}$$

where $E(a, Q)$ is a given ellipsoid in n-space, Σ is the class of C matrices $n \times n$ whose elements c_{ij} satisfy the inequalities (b_{ij} are given non-negative numbers)

$$|c_{ij}| \leqslant b_{ij}, \quad i, \ j = 1, \ \ldots, \ n \tag{3.5}$$

An admissible solution of Problem 3 is proposed in the next section.

Finally, the third operation (addition of ellipsoids) was carried out in [1–3], where the construction yielded resultant ellipsoids of optimum volume. A more general optimum criterion was considered in [2, 3, 5].

We will present the results for the case when the resultant ellipsoid is of minimum volume. Recall that volume as an optimum criterion has the special property that an ellipsoid of optimum volume is invariant under linear transformations: it remains optimal even if subjected to a linear transformation together with the initial ellipsoids that contain the vectors x_1 and x_2.

Let $x_i \in E(a_i, Q_i)$ ($i = 1, 2$), where one of the matrices Q_1 or Q_2 may be singular (for example, let Q_1 be positive definite and Q_2 only positive semidefinite). Then the parameters of the ellipsoid of least volume $E\,(a^+, Q^+)$ containing the sum x_1 and x_2 are determined by the relations

$$a^+ = a_1 + a_2, \quad Q^+ = (p^{-1}+1)Q_1 + (p+1)Q_2 \tag{3.6}$$

where $p > 0$ is the unique positive root of the equation

$$\sum_{j=1}^{n} \frac{1}{p+\lambda_j} = \frac{n}{p(p+1)} \tag{3.7}$$

The numbers $\lambda_j \geqslant 0$ ($j = 1, \ldots, n$) are the roots of the characteristic equation

$$\det(Q_1 - \lambda Q_2) = 0 \tag{3.8}$$

each root being counted according to its multiplicity.

Thus, the construction of the ellipsoid $E(a(t_{i+1}), Q(t_{i+1}))$, ,and hence the solution of Problem 1, will be complete, provided we can solve the auxiliary Problem 3.

4. SOLUTION OF THE AUXILIARY PROBLEM

We note some properties of the set Ω defined by (3.4). The set Ω is star-like relative to the origin, though it need not be convex; it is symmetric relative to all the coordinate hyperplanes.

To prove that the set is star-like, take any point $y = Cx$ of Ω, corresponding to some matrix $C \in \Sigma$ and point $x \in E(a, Q)$. Since the elements c_{ij} of C satisfy inequalities (3.35), it follows that the same is true of the elements αc_{ij} of the matrix αC, where $\alpha \in [0, 1]$. Consequently, the point $y = \alpha Cx$ lies in Ω, and so Ω is star-like.

Here is an example of a non-convex set Ω. Let $n = 2$ and let the constants b_{ij} in the constraints (3.5) be

$$b_{11} = b_{22} = 1, \quad b_{12} = b_{21} = 0 \tag{4.1}$$

In this case, each point $x = (x_1, x_2)$ of the ellipse $E(a, Q)$ is associated through (3.4), (3.5) and (4.1) with the rectangle

$$|y_1| \leqslant |x_1|, \quad |y_2| \leqslant |x_2| \tag{4.2}$$

which is a subset of Ω. In fact, the set Ω itself is the union of rectangles of the form (4.2), where the point $x = (x_1, x_2)$ runs over the ellipse $E(a, Q)$. In particular, suppose that $E(a, Q)$ is degenerate, that is, a segment P_1P_2 in the x_1x_2 plane, where P_1 lies in the second and P_2 in the fourth quadrant (Fig. 1). Suppose that the segment P_1P_2 does not pass through the origin. Then the set Ω in the y_1y_2 plane is a polygon, not necessarily convex, but symmetrical about both coordinate axes. Its boundaries, shown by the thick line in Fig. 1, consist of parts of the boundaries of the rectangles (4.2) corresponding to the points P_1 and P_2, and possibly part of the segment P_1P_2 itself and its mirror images in the coordinate exes.

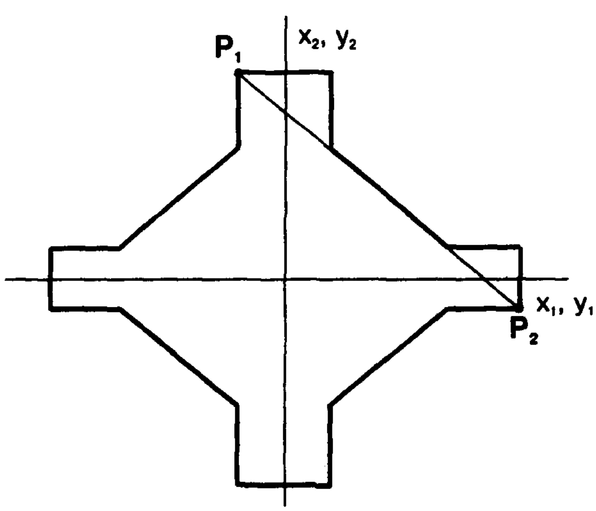

Fig. 1.

We will now prove that Ω is symmetric with respect to all coordinate hyperplanes $y_i = 0$ ($i = 1, \ldots, n$). Take any point $y = Cx \in \Omega$ corresponding to some $C \in \Sigma$ and $x \in E(a, Q)$. Switching signs in a whole row c_{ij} ($j = 1, \ldots, n$) of C, we obtain a matrix C' satisfying conditions (3.5), i.e. $C' \in \Sigma$. Consequently, the point $y = C'x$, which differs from y only in the sign of y_i, is also an element of Ω. This proves the symmetry property of Ω.

Remark 1. Suppose that for some i we have $b_{ij} = 0$ for all $j = 1, \ldots, n$. Then by (3.4) and (3.5) we obtain $y_i = 0$, i.e. in that case Ω lies in the hyperplane $y_i = 0$. This is the case if C has no indeterminacy in certain rows.

Proceeding now to the solution of Problem 3, we first construct a rectangular parallelepiped

$$|y_k| \leq y_k^*, \quad k = 1, \ldots \tag{4.3}$$

containing the set Ω of (3.4). Setting $x = a + \xi$, we obtain, using (3.4) and (3.5)

$$|y_k| \leq \left|\sum_{j=1}^{n} c_{kj} a_j\right| + \left|\sum_{j=1}^{n} c_{kj} \xi_j\right| \leq \sum_{j=1}^{n} b_{kj} |a_j| + \max_{C \in \Sigma} \max_{\xi \in E(0,Q)} |(c^k, \xi)| \tag{4.4}$$

where c^k is the n-vector with components c_{kj} ($j = 1, \ldots, n$). To compute the last maximum in (4.4), we define a Lagrange function

$$L = (c^k, \xi) + \lambda(Q^{-1}\xi, \xi)$$

and equate its gradient with respect to ξ to zero. This gives

$$\xi = -(2\lambda)^{-1} Q c^k \tag{4.5}$$

Substituting (4.5) into the condition $(Q^{-1}\xi, \xi) = 1$, we obtain an equality from which we can determine λ, apart from the sign

$$\lambda = \pm(1/2)(Qc^k, c^k)^{1/2}$$

Using this expression for λ, we deduce from (4.5) that

$$\xi = \mp(Qc^k, c^k)^{-1/2} Q c^k$$

Substituting this expression into (c^k, ξ), we find the required maximum in (4.4)

$$\max_{\xi \in E(0,Q)} |(c^k, \xi)| = (Qc^k, c^k)^{1/2} = \left(\sum_{p,q=1}^{n} Q_{pq} c_{kp} c_{kq}\right)^{1/2} \tag{4.6}$$

Now, in accordance with (4.4), we must find the maximum of (4.6) over $C \in \Sigma$, that is, over all c_{ij} satisfying conditions (3.5). Since the quadratic form (Qc^k, c^k) is a convex function of c^k, it follows that the maximum is reached at a vertex of the parallelepiped defined by inequalities (3.5). We have

$$c_{ij} = b_{ij}\sigma_{ij}, \quad \sigma_{ij} = \pm 1, \quad i, \ j = 1, \ldots, n \tag{4.7}$$

Taking account of (4.4), (4.6) and (4.7), we can write the expression for y_k^* of (4.3) as

$$y_k^* = \sum_{j=1}^{n} b_{kj} |a_j| + \left(\max_{\sigma} \sum_{p,q=1}^{n} Q_{pq} b_{kp} b_{kq} \sigma_{kp} \sigma_{kq}\right)^{1/2} \tag{4.8}$$

where the maximum is considered over all $\sigma_{ij} = \pm 1$ ($j = 1, \ldots, n$).

Remark 2. Formulae (4.8) yield upper bounds for the dimensions of a parallelepiped (4.3) containing Ω. These bounds need not be optimal (they may be lowered).

Remark 3. The actual calculation of the maximum in (4.8) requires scanning 2^{m-1} different cases, corresponding to the different signs of σ_{ij}, where m is the number of non-vanishing elements b_{ij}; the exponent is $m - 1$ because simultaneous change of all signs of σ_{ij} does not change the sum to be maximized in (4.8), and therefore the sign of one of the numbers σ_{ij} may be fixed at will. In the general case we have $m = n^2$, but in many applications the perturbations or indeterminacy are in fact present for only some of the elements of C, and then m is not large. But if only one element of C is indeterminate, that is, all the b_{ij}s vanish except for one $b_{\varkappa p} > 0$, then $m = 1$ and there is no need at all to scan. In that case formula (4.8) is simplified to

$$y_k^* = b_{kp}(|a_p|+Q_{pp}^{1/2}), \quad y_i^* = 0, \quad i \neq k \tag{4.9}$$

Let us construct an ellipsoid containing the parallelepiped (4.3). Since the axes of the parallelepiped coincide with the coordinate exes, the required ellipsoid may be taken as

$$\sum_{k=1}^{n} r_k^{-2} y_k^2 \leqslant 1 \tag{4.10}$$

The lengths of the semi-axes r_k are chosen so that the ellipsoid (4.10) should contain the parallelepiped (4.3). This leads to the condition

$$\sum_{k=1}^{n} r_k^{-2} (y_k^*)^2 = 1 \tag{4.11}$$

In addition, we require that when condition (4.11) is satisfied the volume of the ellipsoid (4.10) should be a minimum. Since the volume of an ellipsoid is proportional to the product of the lengths of its semi-axes r_j, this conditional extremum problem is associated with the Lagrange function

$$L_1 = \prod_{k=1}^{n} r_k + \lambda_1 \sum_{k=1}^{n} r_k^{-2} (y_k^*)^2$$

Equating the derivatives of this function with respect to r_k to zero, we obtain

$$r_k^2 = \lambda_2 (y_k^*)^2, \quad \lambda_2 = 2\lambda_1 \left(\prod_{k=1}^{n} r_k \right)^{-1}, \quad k = 1, \ldots, n$$

Substituting this expression for r_k^2 into (4.11), we obtain $\lambda_2 = n$. Hence

$$r_k = n^{1/2} y_k^*, \quad k = 1, \ldots, n \tag{4.12}$$

This result may be slightly improved if for some i it is true that $b_{ij} = 0$ for all $j = 1, \ldots, n$. In that case, Remark 1 implies that for those values of i we have $y_i^* = 0$. Suppose that the number of indices i such that $b_{ij} = 0$ for all $j = 1, \ldots, n$ is ν, $1 \leqslant \nu < n$. In that case the set Ω lies in an $(n - \nu)$-dimensional hyperplane and it is reasonable to approximate it by an $(n - \nu)$-dimensional ellipsoid in the same hyperplane, of minimal $(n - \nu)$-dimensional volume. Therefore, instead of (4.12), we obtain

$$r_k = (n - \nu)^{1/2} y_k^*, \quad k = 1, \ldots, n \tag{4.13}$$

These results may be summed up in a theorem that presents the solution of Problem 3.

Theorem 1. The set Ω of (3.4) is contained in an ellipsoid (4.10) whose semi-axes r_k are defined by (4.13) and (4.8), that is

$$\Omega \subset E(0, R), \quad R = \operatorname{diag}(r_1^2, \ldots, r_n^2)$$

$$r_k^2 = (n - \nu)\left[\sum_{j=1}^{n} b_{kj} |a_j| + \left(\max_{\sigma} \sum_{p,q=1}^{n} Q_{pq} b_{kp} b_{kq} \sigma_{kp} \sigma_{kq} \right)^{1/2} \right]^2 \quad k = 1, \ldots, n \tag{4.14}$$

The maximum is taken over all $\sigma_{ij} = \pm 1$ $(i, j = 1, \ldots, n)$ such that $b_{ij} \neq 0$, and ν is the number of indices i for each of which $b_{ij} = 0$ $(j = 1, \ldots, n)$.

Note that although the ellipsoid (4.10) may have semi-axes of zero length and the matrix R^{-1} is then undefined, the notation $E(0, R)$ remains meaningful and may be used.

5. APPROXIMATION OF ATTAINABILITY SETS FOR DISCRETE TIME SYSTEMS

Summing up the results of Sections 3 and 4, we can present the solution of Problem 1. To that end we describe the procedure for constructing the functions $a(t_i)$, $Q(t_i)$ occurring in (1.7). Choose some ellipsoid $E(a_0, Q_0)$ containing the initial set M (for example, an ellipsoid of least volume), so that $M \subset E(a_0, Q_0)$, and put

$$a(t_0)=a_0, \quad Q(t_0)=Q_0 \tag{5.1}$$

The vector $a(t_{i+1})$ and the matrix $Q(t_{i+1})$ are expressed in terms of the quantities $a(t_i)$, $Q(t_i)$ ($i = 0, 1, \ldots$) using recurrent relations that follow from (3.6), (3.3) and (4.14). In the notation of (3.6), we deduce from (3.3) and (4.14) that

$$a^+ = a_1 = C_0(t_i)a(t_i)+f(t_i), \quad a_2=0$$

and so

$$a(t_{i+1})=C_0(t_i)a(t_i)+f(t_i) \tag{5.2}$$

Similarly, we deduce from (3.6), (3.3) and (4.14) that

$$Q(t_{i+1})=(p^{-1}+1)Q_1+(p+1)Q_2, \tag{5.3}$$

$$Q_1=C_0(t_i)Q(t_i)C_0^T(t_i), \quad Q_2=R$$

The diagonal matrix R is defined by (4.14) after the substitution

$$a=a(t_i), \quad Q(t_i), \quad b_{jk}=b_{jk}(t_i) \tag{5.4}$$

The scalar parameter p in (5.3) is determined from formulae (3.7) and (3.8), as described at the end of Section 3.

We state the result obtained as follows.

Theorem 2. The recurrent procedure for constructing the parameters of the ellipsoids $E(a(t_i), Q(t_i))$, $i = 0, 1, \ldots$, as determined by (5.1)–(5.4) and formulae (4.14), (3.7) and (3.8), yields a solution of Problem 1.

Remark 4. By the recurrent nature of the construction, the approximating ellipsoids obtained here possess the property of super-attainability [2, 3], which is analogous to the evolution property (1.5) of attainability sets

$$E(a(t_i), Q(t_i)) \supset D(t_i, t_j, E(a(t_j), Q(t_j))), \quad 0 \leqslant j \leqslant i \tag{5.5}$$

6. APPROXIMATION OF ATTAINABILITY SETS FOR CONTINUOUS TIME SYSTEMS

A continuous time system (2.1) may be treated as a limiting case of a discrete time system (1.1). The simplest finite-difference approximation of system (2.1), (2.2) may be written as

$$x(t+\Delta) \approx x_1 + x_2 \tag{6.1}$$

$$x_1=[I+\Delta C_0(t)]x(t)+\Delta f(t), \quad x_2=\Delta C_1(t)x(t)$$

where Δ is a sufficiently small time step and I is the $n \times n$ identity matrix.

Relations (6.1) are analogous to (3.1), and therefore equalities (5.2) and (5.3) can be used, with the obvious changes in notation. By (5.2), we have

$$a(t+\Delta) = [I + \Delta C_0(t)]a(t) + \Delta f(t)$$

Dividing this equality by Δ and letting $\Delta \to 0$, we obtain a differential equation for $a(t)$

$$\dot{a} = C_0(t)a + f(t) \tag{6.2}$$

Comparing (6.1) and (3.1), we obtain, by (5.3)

$$Q(t+\Delta) = (p^{-1}+1)Q_1 + (p+1)Q_2 \tag{6.3}$$

$$Q_1 = [I + \Delta C_0(t)]Q(t)[I + \Delta C_0^T(t)], \quad Q_2 = R$$

The matrix R is defined by (4.14). The roles of the vector a and the matrix Q in (4.14) are now played by the parameters of the ellipsoid $E(a(t), Q(t))$ for the vector $x(t)$, and the roles of the elements b_{ij} by the corresponding bounds $\Delta b_{ij}(t)$ for the absolute values of the elements of the matrix $\Delta C_1(t)$ by which $x(t)$ is multiplied in (6.1). Thus, we deduce from (4.14) that the matrix R may be written as

$$R = \Delta^2 G\,(t) \tag{6.4}$$

The matrix G is given by the previous formula (4.14) for R, that is

$$G = \operatorname{diag}\left\{(n-\nu)\left[\sum_{j=1}^{n} b_{kj}|a_j| + \left(\max_{\sigma} \sum_{p,q=1}^{n} Q_{pq} b_{kp} b_{kq} \sigma_{kp} \sigma_{kq}\right)^{1/2}\right]^2\right\} \tag{6.5}$$

in which we must set $a = a(t)$, $Q = Q(t)$, $b_{ij} = b_{ij}(t)$ $(i, j = 1, \ldots, n)$. Here, as in (4.8) and (4.14), the maximum is taken over all $\sigma_{ij} = \pm 1$ for which $b_{ij} > 0$ $(i, j = 1, \ldots, n)$, and ν is the number of indices k for each of which $b_{kj} = 0$ for all $j = 1, \ldots, n$.

In view of (6.3) and (6.4), the characteristic equation (3.8) may be written as

$$\det[Q(t) + Q(\Delta) - \Delta^2 \lambda G(t)] = 0 \tag{6.6}$$

The roots λ_j of the characteristic equation (6.6) as $\Delta \to 0$ will be sought in the form

$$\lambda_j = \Delta^{-2}(\mu_j)^{-1} + \ldots, \quad j = 1, \ \ldots, \ n \tag{6.7}$$

where μ_i are new unknowns and the dots denote quantities of higher order of smallness in Δ. Substituting (6.7) into (6.6) we obtain after reduction an equation for μ_j

$$\det[Q^{-1}(t)G(t) - \mu_j I] = 0 \tag{6.8}$$

Since Q is positive definite and G is positive semidefinite, this equation has n non-negative roots μ_j, counting each root in accordance with its multiplicity.

We will seek the unique positive root of Eq. (3.7) in the form

$$p = \Delta^{-1} q^{-1} + \ldots \tag{6.9}$$

Substituting (6.7) and (6.9) into Eq. (3.7), expanding both sides in powers of Δ and comparing coefficients of Δ^2, we obtain

$$q = n^{-1/2}\left(\sum_{j=1}^{n} \mu_j\right)^{1/2} \tag{6.10}$$

The sum of the roots of the characteristic equation (6.8) is equal to the trace of the matrix $Q^{-1}G$.

Consequently, it follows from (6.10) that

$$q = \{n^{-1}[\mathrm{Tr}(Q^{-1}G)]\}^{1/2} \tag{6.11}$$

We substitute formulae (6.4) and (6.9) into (6.3) for $Q(t + \Delta)$ and transform, dropping terms of order Δ^2 and higher. Dividing the resulting equality by Δ and letting $\Delta \to 0$, we obtain

$$Q^{\cdot} = C_0(t)Q + QC_0^T(t) + qQ + q^{-1}G \tag{6.12}$$

Equations (6.2) and (6.12), together with formulae (6.5) for G and (6.11) for q, constitute a system of differential equations of order $n + n(n + 1)/2$ for the vector $a(t)$ and the symmetric positive definite matrix $Q(t)$. To obtain initial conditions, as in Section 5, we construct an ellipsoid $E(a_0, Q_0)$ containing the initial set M of (2.1) ($E(a_0, Q_0) \supset M$), and set

$$a(s) = a_0, \quad Q(s) = Q_0 \tag{6.13}$$

We sum up the results as a theorem presenting the solution of Problem 2.

Theorem 3. The ellipsoid $E(a(t), Q(t))$ whose parameters $a(t)$ and $Q(t)$ are the solution of the initial-value problem for the system of differential equations (6.2) and (6.12), taking into consideration that (6.5) holds for G and (6.11) for q, with initial conditions (6.13), satisfies the inclusion (2.5) for $t \geqslant s$.

Remark 5. The linear system (6.2) for the vector $a(t)$ may be integrated independently of the non-linear system (6.12) for the matrix $Q(t)$. The latter system, however, depends on (6.2), since its right-hand sides involve the vector $a(t)$ (see (6.5)).

Remark 6. The approximating ellipsoids we have constructed possess a super-attainability property [2, 3] similar to (5.5)

$$E(a(t), Q(t)) \supset D(t, \tau, E(a(\tau), Q(\tau)))), \quad 0 \leqslant \tau \leqslant t$$

This property, which is analogous to the evolution property (2.4) for attainability sets, follows from the construction procedure itself: the ellipsoids are constructed at any subsequent time on the basis of the ellipsoids at the preceding times.

Remark 7. The non-linear system (6.12) for Q is similar to the analogous system for the case of additively occurring perturbations [1–3]. The difference lies in expression (6.5) for the matrix G which, as already pointed out, depends on the vector a and involves maximization over $\sigma_{ij} = \pm 1$.

Remark 8. The maximum operation over σ_{ij} in (6.5) is simplified considerably if the number m of actually indeterminate parameters c_{ij} is small, and particularly if $m = 1$, when one has just one perturbed (indeterminate) element c_{kp} of the matrix C. In that case, as in the case of (4.9), we deduce from (6.5) (necessarily putting $\nu = n - 1$)

$$G = \mathrm{diag}(0, \ldots, G_k, \ldots, 0), \quad G_k = b_{kp}^2(|a_p| + Q_{pp}^{1/2})^2 \tag{6.14}$$

Remark 9. The outer ellipsoidal approximations we have constructed for attainability sets are not optimal, but at some points in the argument (formulae (3.6)–(3.8) and (4.14)) we have used procedures that are optimal in the sense of the volume of the approximating ellipsoids. Instead of these procedures, one can use other relationships, such as optimal procedures in the sense of the sum of squares of the semi-axes, but then the invariance property would be lost.

7. EXAMPLE

Consider the two-dimensional system ($n = 2$)

$$x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -x_1 + c(t)x_1, \quad |c(t)| \leqslant b \tag{7.1}$$

where $c(t)$ is an undefined bounded perturbation and b is a positive constant. If $c(t)$ is a periodic function, system (7.1) describes parametric excitation of oscillations. In the case of system (7.1), there is only one non-zero element c_{21} of the matrix C and formula (6.14) is applicable with $k = 2, p = 1$. We have

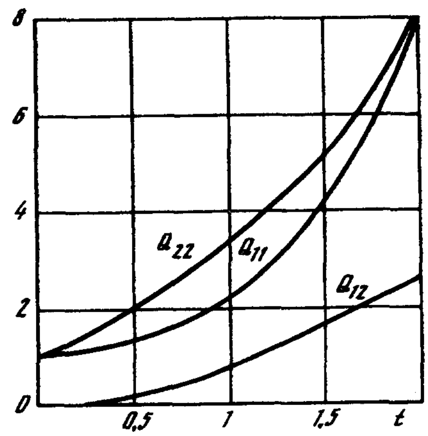

Fig. 2.

$$C_0 = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad G = \begin{Vmatrix} 0 & 0 \\ 0 & b^2(|a_1| + Q_{11}^{1/2})^2 \end{Vmatrix} \tag{7.2}$$

System (6.2) becomes

$$\dot{a}_1 = a_2, \quad \dot{a}_2 = -a_1 \tag{7.3}$$

and describes harmonic oscillations.

We compute q from (6.11) and from system (6.12), taking (7.2) into consideration

$$\dot{Q}_{11} = 2Q_{12} + qQ_{11}, \quad \dot{Q}_{12} = Q_{11} - Q_{22} + qQ_{12} \tag{7.4}$$

$$\dot{Q}_{22} = -2Q_{12} + qQ_{22} + q^{-1}b^2(|a_1| + Q_{11}^{1/2})^2$$

$$q = bQ_{11}^{1/2}(|a_1| + Q_{11}^{1/2})D^{-1}, \quad D = [2(Q_{11}Q_{22} - Q_{12}^2)]^{1/2}$$

Suppose that the initial set M at time $t = 0$ is a disk of radius ε in the x_1x_2 plane with centre at the origin. Then

$$a_1(0) = a_2(0) = 0, \quad Q_{11}(0) = Q_{22}(0) = \varepsilon^2, \quad Q_{12}(0) = 0 \tag{7.5}$$

System (7.3) with initial conditions (7.5) has a trivial solution, and system (7.4) then becomes

$$\dot{Q}_{11} = 2Q_{12} + bQ_{11}^2 D^{-1}, \quad \dot{Q}_{12} = Q_{22} - Q_{11} + bQ_{11}Q_{12}D^{-1} \tag{7.6}$$

$$\dot{Q}_{22} = -2Q_{12} + bQ_{11}Q_{22}D^{-1} + bD$$

The right-hand sides of system (7.6) are homogeneous functions of Q_{ij}: the system is invariant under the transformation $Q_{ij} \to \lambda Q_{ij}$ with parameter λ. Therefore, we may assume without loss of generality that $\varepsilon = 1$ in (7.5). The results of a numerical solution of the initial-value problem (7.6), (7.5) with $b = 0.8$ are shown in Fig. 2. To interpret these results, we recall that the support function of the ellipse $E(0, Q(t))$ is $\rho(z) = (Qz, z)^{1/2}$, and therefore the quantities $Q_{11}^{1/2}$, $Q_{22}^{1/2}$ may be interpreted as the projections of the ellipse $E(0, Q(t))$ on to the x_1 and x_2 axes, respectively. Hence the following estimates hold for all solutions of the initial system (7.1) with the initial conditions (7.5)

$$|x_i(t)| \leq [Q_{ii}(t)]^{1/2}, \quad i = 1, 2$$

The outer approximations obtained in this paper for attainability sets may be useful when one is estimating the influence of perturbations affecting the matrix of the system.

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